PHYSICAL REVIEW E **67**, 045203(R) (2003)

Universal scaling of Lyapunov exponents in coupled chaotic oscillators

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(Received 2 October 2002; published 28 April 2003)

We have uncovered a phenomenon in coupled chaotic oscillators where a subset of Lyapunov exponents, which are originally zero in the absence of coupling, can become positive as the coupling is increased. This occurs for chaotic attractors having multiple scrolls, such as the Lorenz attractor. We argue that the phenomenon is due to the disturbance to the relative frequencies with which a trajectory visits different scrolls of the attractor. An algebraic scaling law is obtained which relates the Lyapunov exponents with the coupling strength. The scaling law appears to be universal.

DOI: 10.1103/PhysRevE.67.045203 PACS number(s): 05.45.Xt, 05.40.−a

The collective dynamics of coupled chaotic oscillators have been a topic of continuous interest in nonlinear and statistical physics. In a typical setting where a group of chaotic oscillators are coupled, one intuitively expects that, as the coupling parameter K is increased from zero, a coherence among the oscillators emerges. Realizing that the dynamics of individual oscillators are completely independent of each other for $K=0$ and hence in this case the collective dynamics of the whole system can be regarded as "complicated," the presence of a finite amount of coherence for $K\neq 0$ means that the collective dynamics is less complicated. Indeed, it has been well documented that chaotic synchronization of different forms can arise as a result of coupling [1–4], which has been one of the most active research areas in nonlinear dynamics [5]. The purpose of this paper is to report our finding of a quite counterintuitive phenomenon: the collective dynamics of coupled chaotic oscillators can become more complicated due to coupling. This phenomenon is expected to be quite general, as it can occur for typical chaotic attractors with multiple scrolls in the phase space, such as the Lorenz attractor [6] and, its occurrence does not seem to depend on the way by which the oscillators are coupled.

To explain our finding, we consider the simple setting of a system of two mutually coupled, three-dimensional chaotic oscillators. The chaotic dynamics of each oscillator is then defined by a positive, a zero, and a negative Lyapunov exponents. For $K=0$, the whole system is six-dimensional with two positive, two zero, and two negative exponents. Current understanding of coupled chaotic systems would suggest the following. As K is increased from zero, one of the zero Lyapunov exponent would become negative (at $K_{PS}>0$), indicating chaotic phase synchronization [4]. As K is increased further, one of the originally positive exponents becomes negative (at $K_{AS}>K_{PS}$), signifying chaotic synchronization in amplitude [2], after which the system possesses only one positive Lyapunov exponent. The behavior of the Lyapunov exponents in these parameter regimes of interest is therefore the following. Prior to phase synchronization ($0<K<K_{PS}$), there are two positive, two zero, and two negative Lyapunov exponents. In the phase-synchronization regime ($K_{PS}<K<K_{AS}$), there are two positive, one null, and three negative exponents, whereas in the amplitude-synchronization regime ($K>K_{AS}$), there are one positive, one null, and four negative exponents. Observe that in these parameter regimes, the number of positive Lyapunov exponents does not exceed two. This scenario has indeed been observed and reported extensively in literature for chaotic attractors such as those from the Rössler system [7]. But can this be true generally?

In this paper, we present evidence that the above seemingly well accepted picture of coupled chaotic dynamics cannot be true, in general. We find that a system of N coupled chaotic attractors with multiple scrolls can exhibit a striking phenomenon in the *weakly* coupling regime: a subset of null Lyapunov exponents for $K=0$ can become *positive* for $K>0$. For a system of N coupled chaotic oscillators, there exists a parameter regime of finite Lebesgue measure: $0<K<K_{PS}$, for which the number of positive Lyapunov exponents is $N+M$, where $M=N/2$ for N even and $M=(N+1)/2$ for N odd. Let $\lambda_1\geqslant\lambda_2\geqslant\cdots\geqslant\lambda_N>\lambda_{N+1}>\cdots>\lambda_{N+M}$ denote the positive part of the Lyapunov spectrum of the system. As K is increased from zero, the additional M exponents appear to obey the following algebraic scaling law:

$$\lambda_j\sim K^{\alpha} \quad \text{for } j=N+1,\ldots,N+M, \tag{1}$$

which is expected to be general [8]. For three-dimensional chaotic attractors, the scaling exponent α assumes the universal value of 2. We will also argue that these results hold, regardless of the coupling scheme (whether local or global, for instance). We believe that these results have not been noticed before but they are important and of broad interest considering that coupled chaotic dynamics have been a continuously pursued topic in nonlinear and statistical physics.

We first present results with the following system of two coupled Lorenz oscillators: $\dot{x}_{1,2}=\sigma_{1,2}(y_{1,2}-x_{1,2})+K_{1,2}(x_{2,1}-x_{1,2})$, $\dot{y}_{1,2}=28x_{1,2}-y_{1,2}-x_{1,2}z_{1,2}$, $\dot{z}_{1,2}=-(8/3)z_{1,2}+x_{1,2}y_{1,2}$, where $\sigma_{1,2}$ are parameters of the Lorenz oscillator that can be set at the same or slightly different values so that the oscillators possess identical or nonidentical chaotic attractors, and $K_{1,2}$ are the coupling parameters associated with

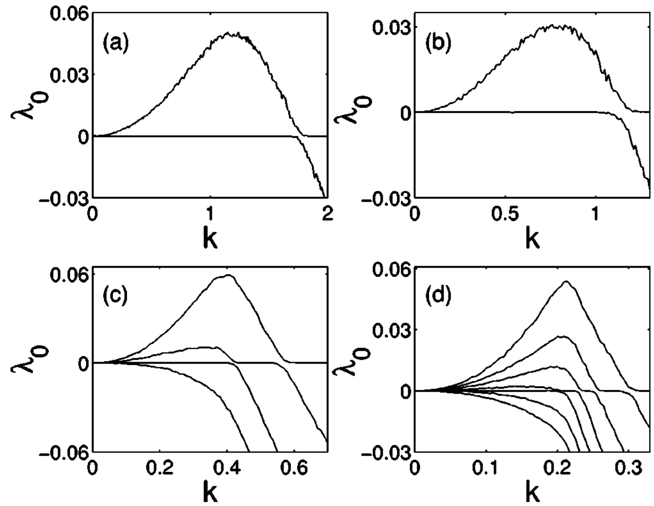

FIG. 1. For the system of coupled Lorenz chaotic oscillators, the originally null Lyapunov exponents versus the coupling parameter K for (a) $N=2$, $\sigma_1=\sigma_2=10$, $K_1=K$, and $K_2=0$; (b) $N=2$, $\sigma_1=10$ but $\sigma_2=11$, $K_1=K$, and $K_2=0$; (c) $N=4$, $\sigma_i=10$ ($i=1,\ldots,4$), $K_i=K$ ($i=1,\ldots,4$); and (d) $N=7$, $\sigma_i=10$ ($i=1,\ldots,7$), $K_i=K$ ($i=1,\ldots,7$).

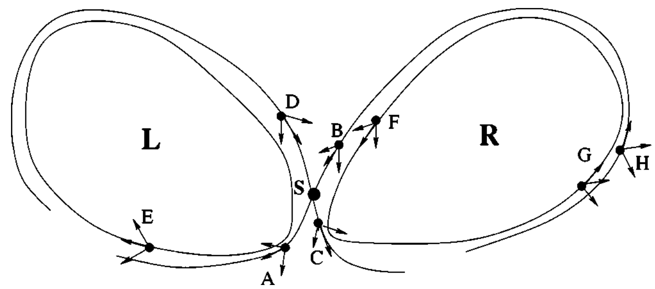

FIG. 2. Schematic illustration of the consistency of a typical double-scroll chaotic flow. Random perturbations that move a trajectory from points B to F, or from G to H, etc., are consistent with the noise-free dynamics. Those that cause the trajectory to go from B to D, or C to A, or vice versa, are inconsistent with the original dynamics.

the two oscillators which can be set so that the coupling scheme is unidirectional or bidirectional. Figure 1(a) shows the two originally null Lyapunov exponents versus the coupling parameter K for $\sigma_1=\sigma_2=10$ (identical oscillators), $K_1=K$ and $K_2=0$ (unidirectional coupling). We observe that one exponent, which is zero for $K=0$, starts to increase as K is increased from zero. The exponent reaches a maximum and then decreases as K is increased further. It becomes negative for $K>K_{PS}$, signifying a coherence in the phases of the two oscillators [4]. Since each Lorenz oscillator possesses one positive Lyapunov exponent in the small range of K values considered, for $0<K<K_{PS}$ the coupled system has *three positive Lyapunov exponents*. Considering that the uncoupled system for $K=0$ has only two positive exponents, we see that the coupled system becomes more complicated in the sense that it has one more positive exponent. We find this behavior robust, as it holds under structural changes in the system. For instance, Fig. 1(b) shows the behavior of the exponents as a function of K for $K_1=K$ and $K_2=0$ (unidirectional coupling), $\sigma_1=10$ but $\sigma=11$ (nonidentical oscillators). Figures 1(c) and 1(d) show the exponents of the system under bidirectional coupling ($K_1=K_2=K$) for $N=4$ and $N=7$ identical coupled oscillators, respectively, indicating a similar behavior for systems with more than two coupled oscillators.

The behavior of the Lyapunov exponents shown in Figs. 1(a–d) is apparently quite different from that, say, of a system of two coupled chaotic Rössler oscillators, where none of the null exponents becomes positive as the coupling is increased. Thus, the distinct behavior shown in Figs. 1(a–d) must come from the double-scroll feature of the Lorenz attractor. As a trajectory moves on the Lorenz attractor and switches back and forth between the two scrolls, there is a fundamental consistency that the trajectory must follow. The coupling, which is chaotic, can destroy this consistency and cause a zero Lyapunov exponent to become positive.

Figure 2 shows, schematically, the topology of the Lorenz chaotic attractor, where A–H are representative points associated with continuous trajectories, and the arrows denote the corresponding local eigenspaces. There are two scrolls, denoted by "L" and "R," respectively. A typical trajectory visits both scrolls intermittently in time. It does so by executing topologically circular motion in one scroll with a number of rotations and switches to another, and so on. The region surrounding the point "S" is where the switches occur. In order to go from "L" to "R," the trajectory has to follow points such as D and C. Without perturbations, it is not possible to go directly from "L" to "R" when the trajectory is near point A. Similarly, in order to go from "R" to "L," the trajectory must follow the path through points B and A. When the trajectory is near point C, it is not possible for it to go directly to "L" without going through at least one rotation in "R." The dynamical consistency so described guarantees the existence of a zero Lyapunov exponent, which corresponds to the neutral direction of the flow, in spite of the intermittent switching behavior. But the presence of irregular perturbation could disturb this consistency, as we will argue below.

Now consider the coupled system. Because it is deterministic, one null Lyapunov exponent must be preserved, despite coupling. The behavior of the second, originally zero Lyapunov exponent can be assessed by regarding the linear coupling term as a randomlike, time-varying perturbation to a single Lorenz oscillator. This is particularly true when the coupling is small, as the flow is chaotic. There can be two types of perturbations, depending on where in the phase space they are applied to the flow, which can have very distinct effects on the flow. The first type can occur everywhere on the attractor, and they disturb the trajectory in a way that is consistent with the perturbation-free dynamics. For instance, the perturbation can cause a trajectory to move from points B to F, or from G to H, and so on. This will have little effect on the eigendirections of the flow, as the structures of the eigenspaces at these nearby points, which are not in the vicinity of the switching point **S**, are consistent. In the asymptotic limit, the effects of these *consistent perturbations* will be averaged out, causing no change in the Lyapunov

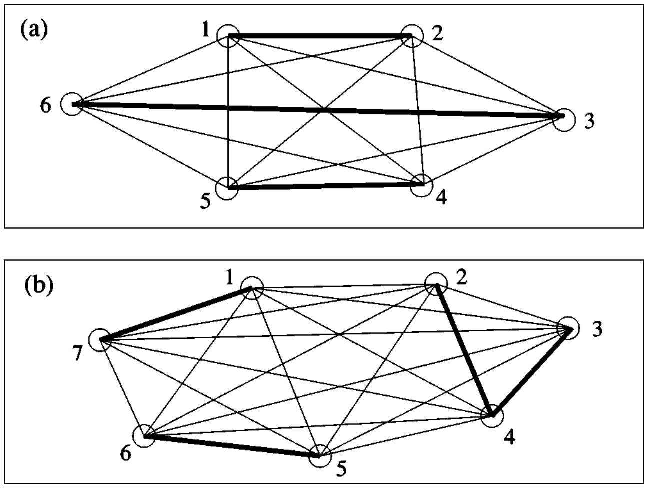

FIG. 3. The minimal number N^P of edges required to cover all N nodes on a graph. (a) For $N=6$ (even), we have $N^P=N/2=3$. (b) For $N=7$ (odd), we have $N^P=(N+1)/2=4$.

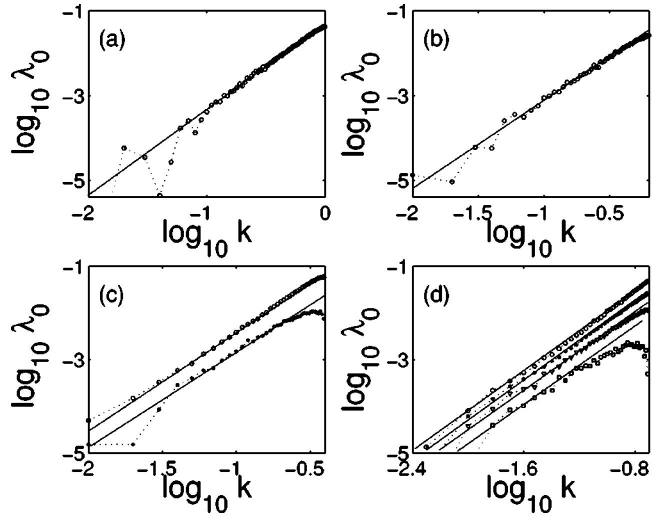

FIG. 4. Test of scaling law (1) under different coupling configurations. (a) $N=2$ (identical oscillators with unidirectional coupling), (b) $N=2$ (nonidentical oscillators with unidirectional coupling), (c) $N=4$ (bidirectional coupling), (d) $N=7$ (bidirectional coupling).

spectrum. This is why the null Lyapunov exponents can be preserved for small coupling in systems such as the coupled Rössler oscillators. The second type of perturbations occurs in the vicinity of **S**, and they are not consistent with the perturbation-free dynamics. For example, such a perturbation can cause a trajectory to go from points B to D, or from A to C, or vice versa. The local eigenspaces associated with these *inconsistent* points are generally quite distinct from each other, as shown in Fig. 2. In particular, the neutral eigendirections associated with two points such as B and D can be quite different. Infinitesimal distance along the neutral direction at point B, when it is moved to point D by the perturbation, can no longer be preserved. Because of the chaotic dynamics, the component of this distance in the unstable direction of point D will be magnified, leading to the destruction of the neutral direction. The associated Lyapunov exponent will then increase from zero as the perturbation is turned on, as we have observed in Fig. 1(a–d). The consequence of this type of perturbation is a change in the relative frequency with which a trajectory visits "L" and "R," which can be measured in numerical experiment. While the discussion above is with respect to the Lorenz attractor, it is clear that it works for other types of chaotic attractors with multiple scrolls.

Is there a universal scaling law governing the increase of the Lyapunov exponent from zero? To answer this question, we use a heuristic approach. For a double-scroll chaotic attractor in three-dimensional phase space, let λ_i^L and λ_i^R ($i=1,2,3$) be the average exponential rates of change of infinitesimal distances along the three eigendirections for a trajectory in the left and right scrolls, respectively. We have $\lambda_1^L>0=\lambda_2^L>\lambda_3^L$ and $\lambda_1^R>0=\lambda_2^R>\lambda_3^R$. We call them *pseudo-Lyapunov exponents* because they can be regarded as arising from nonattracting chaotic sets that give rise to transient chaos [9]. In the perturbation-free case, the null Lyapunov exponent can be trivially expressed as $0=\lambda_2=f^L\lambda_2^L+f^R\lambda_2^R$, where f^L and f^R ($f^L+f^R=1$) are the frequencies of visits to the left and right scrolls, respectively. In the presence of a small coupling of strength K, the frequencies f^L and f^R are disturbed due to the perturbations that cause inconsistencies to the flow. Approximately, we can write

$$\lambda_2(K)\approx f^L(K)\lambda_2^L+f^R(K)\lambda_2^R+f^S(K)\overline{\lambda}=f^S(K)\overline{\lambda}, \quad (2)$$

where $f^S(K)$ is the frequency of switchings caused by the inconsistent perturbations $[f^L(K)+f^R(K)+f^S(K)=1]$, and $\overline{\lambda}>0$ is the average expansion rate of infinitesimal vectors near the switching point **S** (Fig. 2). Suppose there is a dominant unstable steady state in the switching region, as for the Lorenz system, $\overline{\lambda}$ is approximately the positive Lyapunov exponent of this steady state. Thus, we see that the dependence of $\lambda_2(K)$ on K is determined by $f^S(K)$, which is the probability for the perturbed trajectory to fall in the stable manifold of the unstable steady state in the switching region **S**. For a three-dimensional flow, for a point in **S** a perturbed trajectory can be found in a sphere of radius proportional to K which is centered at the point, and the dimension of the stable manifold of the unstable steady state is two [10]. Thus, we have $f^S(K)\sim K^2$, which is the scaling law (1) with the algebraic scaling exponent of two for systems of coupled, three-dimensional chaotic flows.

While the above argument is for a system of two coupled chaotic oscillators, the picture of one chaotic attractor under irregular perturbations is valid for systems of arbitrary numbers of coupled, identical or nonidentical attractors. Consider a system that consists of N ($N>2$) coupled chaotic attractors, which possesses N null Lyapunov exponent when uncoupled. As the coupling is turned on, we expect a subset $N^P<N$ of these exponents to become positive. The question is, what determines N^P, in general? The idea of chaotic attractor under perturbation again provides an answer. In the language of graph theory [11], a system of N coupled oscillators can be viewed as a graph with N nodes and a number

of edges. The manner by which the edges are distributed is determined by the way of coupling, such as global versus local. The number of null Lyapunov exponents that can become positive is then determined by the number of *independent* coupling terms. The question becomes the following on a graph: given N nodes, what is the minimal number of edges that can cover all N nodes? The answer is $N/2$ for N even and $(N+1)/2$ for N odd, as shown schematically in Fig. 3. Our heuristic theory suggests that all the null N^P exponents follow the scaling law (1) to become positive as a function of K.

We now provide numerical support for the scaling law (1). We study the following system of N globally coupled Lorenz chaotic attractors: $dx_i/dt=\sigma_i(y_i-x_i)+\Sigma_{j\neq i}K_{i,j}(x_j-x_i)$, $dy_i/dt=28x_i-y_i-x_iz_i$, and $dz_i/dt=-(8/3)z_i+x_iy_i$. Figures 1(c,d) show, for N even $(N=4)$ and N odd $(N=7)$, respectively, the behaviors of the originally null Lyapunov exponents. We observe that the numbers of these exponents that can become positive under coupling are $N/2$ for N even and $(N+1)/2$ for N odd, confirming our analysis. Furthermore, the scaling of these exponents with the coupling parameter K appears to be algebraic with the universal exponent two, as shown in Figs. 4(a–d), the plots on a logarithmic scale corresponding to Figs. 1(c,d), respectively. While these results are for the global coupling scheme, we obtained similar results for the nearest-neighbor type of coupling. In this case, there are only N edges in Fig. 3 and the number of independent coupling terms is the minimally necessary number of edges that can connect all nodes. As in the case of global coupling, each independent coupling term can cause a null Lyapunov exponent to become positive as the coupling strength is increased from zero. We find that the scaling result appears to hold for other coupling schemes such as multiplicative coupling [12].

In summary, we have uncovered a general phenomenon in coupled chaotic systems. When N chaotic oscillators with double-scroll (or multiscroll) attractors are coupled, $N/2$ (N even) or $(N+1)/2$ (N odd) Lyapunov exponents, which are zero in the absence of coupling, can become positive as soon as the coupling parameter is increased from zero. The way that these exponents become positive from zero can be characterized by a universal algebraic scaling law. These results hold for all cases that we have examined, which include different number of oscillators, different coupling schemes, and various combinations. We believe that our finding is important in chaotic dynamics and that it can be tested experimentally [12].

This work was supported by AFOSR under Grant No. F49620-98-1-0400.

[1] H. Fujisaka and T. Yamada, Prog. Theor. Phys. **69**, 32 (1983); V.S. Afraimovich, N.N. Verichev, and M.I. Rabinovich, M. I. Radio Phys. Quantum electron. **29**, 747 (1986).
[2] L.M. Pecora and T.L. Carroll, Phys. Rev. Lett. **64**, 821 (1990).
[3] L. Kocarev and U. Parlitz, Phys. Rev. Lett. **76**, 1816 (1996).
[4] M.G. Rosenblum, A.S. Pikovsky, and J. Kurths, Phys. Rev. Lett. **76**, 1804 (1996).
[5] L.M. Pecora, T.L. Carroll, G.A. Johnson, D.J. Mar, and J.F. Heagy, Chaos **7**, 520 (1997).
[6] E.N. Lorenz, J. Atmos. Sci. **20**, 130 (1963).
[7] O.E. Rössler, Phys. Lett. **57A**, 397 (1976).
[8] Some pioneering studies on the scaling behavior of the largest Lyapunov exponent in coupled map lattices can be found, for example, in R. Livi, A. Politi, and S. Ruffo, J. Phys. A **25**, 4813 (1992); W. Yang, E.-J. Ding, and M. Ding, Phys. Rev. Lett. **76**, 1808 (1996); A. Torcini, R. Livi, A. Politi, and S. Ruffo, *ibid.* **78**, 1391 (1997).
[9] Typical trajectories restricted to "L" or "R" are transiently chaotic. It has been known that nonattracting chaotic saddles, which are responsible for transient chaos, are the "skeleton" of a chaotic attractor [K.G. Szabo and T. Tél, Phys. Rev. E **50**, 1070 (1994)]. The pseudo-Lyapunov exponents can thus be defined with respect to the conditional measure for transient chaos [T. Tél, in *Directions in Chaos*, edited by B.-L. Hao (World Scientific, Singapore, 1990), Vol. 3].
[10] J. Guckenheimer and P. Holmes, *Nonlinear Oscillations, Dynamical Systems, and Bifurcations of Vector Fields* (Springer, New York, 1983).
[11] R. Diestel, *Graph Theory* (Springer, New York, 2000).
[12] In a future paper, we will give a detailed account of our physical scaling theory and provide extensive numerical check for the scaling law covering a wide array of coupling configurations. We also believe that results in this paper can be tested in experiments using chaotic electronic circuits.